Lecture 11

Last time : $(V, \langle , \rangle)$ Euclidean sp. of dim $m = 2k$

J complex str. $\quad J^2 = -\mathrm{Id}$

$$\Rightarrow \quad V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} = \underset{\sqrt{-1}}{W} \oplus \underset{-\sqrt{-1}}{\overline{W}}$$

$$S = \Lambda^{\bullet} \overline{W}^{*}$$

we defined $\quad C(V) \circlearrowright S \quad$ $\mathbb{Z}_2$-graded Clifford module

so that $\quad \mathrm{End}(S) \simeq C(V) \otimes_{\mathbb{R}} \mathbb{C}$

Prop : When $\dim V = m = 2k$

$\exists !$ nontrivial complex irreducible representation S s.t.

$$\mathrm{End}(S) \simeq C(V) \otimes_{\mathbb{R}} \mathbb{C}$$

$\dim_{\mathbb{C}} S = 2^k$

Moreover, $\exists !$ Hermitian metric h^S on S (up to a scalar multiplication) s.t.

$$\forall \ v \in V, \quad s_1, s_2 \in S$$

$$h^S(c(v)s_1, s_2) = -h^S(s_1, c(v)s_2)$$

Hence, for $v \in V \quad \|v\| = 1$

$$h^S(c(v)s_1, c(v)s_2) = h^S(s_1, s_2) \qquad \#$$

Def : S is called spinor space

elements in S are called spinors.

Now we fix an orientation for V given by the ordered ONB $e_1, \dots, e_{2k}$

Put
$$\tau = (\sqrt{-1})^k c(e_1) \cdots c(e_{2k}) \in \mathrm{End}(S)$$

It is easy to verify $\tau^2 = \mathrm{Id}_S$

for $v \in V$ $\quad \tau c(v) = - c(v) \tau$

τ defines a $\mathbb{Z}_2$-grading on S

$$S = S^+ \oplus S^-$$
$$\tau = 1 \qquad -1$$

$\frac{\sqrt{-1}}{2}(w_j^* \iota_{\bar{w}_j} - \iota_{\bar{w}_j} w_j^*)$

If the orientation is given by $V \otimes_{\mathbb{R}} \mathbb{C} = W \oplus \overline{W}$

that is $\quad e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \bar{w}_j) \quad e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \bar{w}_j)$

$$\Rightarrow \quad S^+ = \Lambda^{\text{even}} \overline{W}^* \qquad S^- = \Lambda^{\text{odd}} \overline{W}^*$$

Prop: If (E, h^E) is a $C(V)$-module s.t. $\forall v \in V$ $c(v)$ is skew-adj. w.r.t h^E, then (E, h^E) is a unitary Representation of $\mathrm{Spin}(V)$.

Theorem: $\mathrm{Spin}(V)$ acts on S preserving $S = S^+ \oplus S^-$

Moreover, S^+, S^- are irreducible unitary representation of $\mathrm{Spin}(V)$.

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HW 4.2: Supertrace of $\mathrm{Spin}(V) \curvearrowright S^\pm$

V. Dirac operator, spin manifold and Lichnerowicz formula

V.1 Dirac operator

Recall: Riemannian geometry

(X, g^{TX}) a Riemannian manifold

$\uparrow$ Riemannian metric

Prop/Def: Levi-Civita connection ∇^{TX} is the unique connection on TX s.t.

① Euclidean: ∇^{TX} preserves g^{TX}

$$d\langle U, V\rangle_{g^{TX}} = \langle \nabla^{TX} U, V\rangle_{g^{TX}} + \langle U, \nabla^{TX} V\rangle_{g^{TX}}$$

② Torsion-free: $\forall\, U, V \in \Gamma(TX)$

$$\nabla^{TX}_U V - \nabla^{TX}_V U = [U, V]$$

$\nwarrow$ Lie bracket

Pf: ∇^{TX} always exists and is determined by the formula: $\forall\, U, V, W \in \Gamma(TX)$

$$\langle \nabla^{TX}_U V, W\rangle_{g^{TX}}$$

$$= \frac{1}{2}\Big\{ U\langle V, W\rangle_{g^{TX}} + V\langle W, U\rangle_{g^{TX}} - W\langle U, V\rangle_{g^{TX}} + \langle [U,V], W\rangle_{g^{TX}} - \langle [V,W], U\rangle_{g^{TX}} - \langle [U,W], V\rangle_{g^{TX}} \Big\}$$

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④

Def: Riemannian curvature tensor R^{TX}

$$R^{TX}(U,V)W = \nabla^{TX}_U \nabla^{TX}_V W - \nabla^{TX}_V \nabla^{TX}_W W - \nabla^{TX}_{[U,V]} W.$$

∇^{T^*X} the connection on T^*X induced by Levi-Civita connection ∇^{TX}

$\longrightarrow$ $\nabla^{\Lambda^\bullet T^*X}$ on $\Lambda^\bullet T^*X$

Prop: ① $d \curvearrowright \Omega^\bullet(X)$ has a formula:
if $\{e_i\}$ is ONB of (TX, g^{TX})

$$d = \sum_i e^i \wedge \nabla^{\Lambda^\bullet T^*X}_{e_i}$$

induced by Levi-Civita connection

② (Cartan formula) For $V \in \Gamma(TX)$

$L_V \curvearrowright \Omega^\bullet(X)$ Lie derivative

$$L_V \alpha := \frac{\partial}{\partial t} \phi_t^* \alpha \qquad \phi_t = \exp(tV)$$

diffeomorphism of X generated by V.

Then $\begin{cases} L_V f = V(f) & \text{for } f \in C^\infty(X) \\ L_V = [d, i_V] & \curvearrowright \Omega^\bullet(X) \end{cases}$

inner product / contraction

pf: HW 4.3 #

Def: (X, g^{TX}) oriented Riemannian mfd of dim n

Riemannian volume form

$$dv(x) := e^1 \wedge \cdots \wedge e^n$$

$\{e_j\}$ orthonormal local frame (TX, g^{TX}) oriented

$dv(x) \in \Omega^n(X)$ and it defines a positive measure on X.

Def: (E, h^E) Hermitian vector bundle on (X, g^{TX}) 5

$\forall\ s_1, s_2 \in C^\infty(X, E)$

$$\langle s_1, s_2 \rangle_{L^2} := \int_X h^E(s_1(x), s_2(x))\, dv(x)$$

$L^2(X, E)$ = completion of $C^\infty(X, E)$ w.r.t. $\langle , \rangle_{L^2}$

separable Hilbert space.

Now we switch to define Dirac operator

(X, g^{TX})

Def: The Clifford module $C(TX)$ is the vector bundle on X s.t. $C(TX)_x = C(T_xX)$ ← w.r.t g^{TX}_x.

Symbol map

$$\sigma : C(TX) \longrightarrow \Lambda^{\cdot} T^*X$$

now is an isomorphism of vector bundles on X

This way, we can equip $C(TX)$ with a Euclidean metric $g^{C(TX)}$ induced from $g^{\Lambda^{\cdot} T^*X}$.

Def: Let $\nabla^{C(TX)}$ be the connection on $C(TX)$ given by

$$(*) \quad \nabla^{C(TX)}_u (v_1 \cdots v_k) = \sum_j v_1 \cdots v_{j-1} (\nabla^{TX}_u v_j) v_{j+1} \cdots v_k$$

$\Leftrightarrow$ induced via σ from $\nabla^{\Lambda^{\cdot} T^*X}$ on $\Lambda^{\cdot} T^*X$.

$$v_j^2 = -\langle v_j, v_j \rangle_{g^{TX}}$$

Since ∇^{TX} preserves g^{TX}

$$\nabla^{TX}_{\omega}(\nu_j^2) = (\nabla^{TX}_{\omega}\nu_j)\nu_j + \nu_j(\nabla^{TX}_{\omega}\nu_j)$$

$$= -2\langle \nabla^{TX}_{\omega}\nu_j, \nu_j\rangle_{g^{TX}}$$

$$= -(\langle \nabla^{TX}_{\omega}\nu_j, \nu_j\rangle_{g^{TX}} + \langle \nu_j, \nabla^{TX}_{\omega}\nu_j\rangle_{g^{TX}})$$

$$= -\omega\langle \nu_j, \nu_j\rangle_{g^{TX}}$$

The definition $(*)$ preserves the Clifford relation.

<u>Lemma</u>: $C(TX)$ acts on $\Lambda^{\bullet}T^*X$ by

$$c(\nu) = \nu^*\wedge - \iota_\nu \quad \text{odd operator}$$

Then $\forall\ \nu \in TX$

$$[\nabla^{\Lambda^{\bullet}T^*X}_{\omega}, c(\nu)] = c(\nabla^{TX}_{\omega}\nu)$$

Pf: $\{e_j\}$ ONB

$$[\nabla^{\Lambda^{\bullet}T^*X}_{\omega}, c(e_j)]\, e^i$$

$$= \nabla^{\Lambda^{\bullet}T^*X}_{\omega}(c(e_j)e^i) - c(e_j)\nabla^{\Lambda^{\bullet}T^*X}_{\omega}e^i$$

$$= \nabla^{\Lambda^{\bullet}T^*X}_{\omega}(e^j\wedge e^i - \delta_{ij}) - e^j\wedge\nabla^{\Lambda^{\bullet}T^*X}_{\omega}e^i$$

$$= (\nabla^{\Lambda^{\bullet}T^*X}_{\omega}e^j)\wedge e^i = c(\nabla^{TX}_{\omega}e_j)\, e^i$$

$$\langle \nabla^{TX}_{\omega}e_j, e_k\rangle = \langle e_j, \nabla^{TX}_{\omega}e_k\rangle$$

$$= \langle \nabla^{\Lambda^{\bullet}T^*X}_{\omega}e^j, e_k\rangle$$

$$c(\nu)(\alpha\wedge\beta) = (c(\nu)\alpha)\wedge\beta + (-1)^{\alpha}\alpha\wedge c(\nu)\beta.$$

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Definition:

① A vector bundle E on X is a Clifford module if $\exists$ a $C(TX)$-action

$$C^\infty(X, C(TX)) \times C^\infty(X, E) \to C^\infty(X, E)$$
$$(a, s) \longmapsto c(a)\cdot s$$

A superbundle $E = E^+ \oplus E^-$ on X is a Clifford module if E is a Clifford module & $\forall v \in TX$ $c(v)$ exchanges E^+ & E^-

$\mathbb{Z}_2$-graded Clifford module

② Let $h^E = h^{E^+} \oplus h^{E^-}$ be Hermitian metric on $\mathbb{Z}_2$-graded Clifford module $E = E^+ \oplus E^-$ on X

Then we call (E, h^E) to be self-adj if

$$\forall v \in TX \quad c(v)^* = -c(v)$$

adj of $c(v)$ w.r.t h^E

③ A connection ∇^E on (E, h^E) is called a Clifford connection if $\forall a \in C^\infty(X, C(TX))$, $s \in C^\infty(X, E)$, $u \in TX$

$$[\nabla^E_u, c(a)]s = c(\nabla^{C(TX)}_u a)s$$

$$\left(\Leftrightarrow \forall u, v \in TX \quad [\nabla^E_u, c(v)] = c(\nabla^{TX}_u v) \right)$$

Rk: ∇^E has to preserve the splitting $E = E^+ \oplus E^-$

i.e. $\nabla^E = \nabla^{E^+} \oplus \nabla^{E^-}$

Ex: $(\Lambda^\bullet T^*X, g^{\Lambda^\bullet T^*X}, \nabla^{\Lambda^\bullet T^*X})$
a self-adj. $\mathbb{Z}_2$-graded Clifford module
with Clifford connection.

Def (Dirac operator)
Given (E, h^E, ∇^E) $\mathbb{Z}_2$-Clifford self-adj. & ∇^E is Hermitian

The Dirac operator D^E associated with ∇^E is defined as

$$D^E := \sum_{j=1}^{m} c(e_j) \nabla^E_{e_j} \quad \circlearrowright \quad C^\infty(X, E)$$

$\{e_j\}_{j=1}^m$ any local orthonormal frame of (TX, g^{TX})

We denote

$$D^E_\pm = D^E \big|_{C^\infty(X, E^\pm)} : C^\infty(X, E^\pm) \longrightarrow C^\infty(X, E^\mp).$$

Example: de Rham-Dirac operator (X, g^{TX}) oriented Riemannian mfd.

$d : \Omega^\bullet(X) \longrightarrow \Omega^{\bullet+1}(X)$

$\langle\ ,\ \rangle_{L^2}$ inner product on $\Omega^\bullet(X)$, $dv(x)$ volume form.

d^* formal adjoint of d w.r.t. $\langle\ ,\ \rangle_{L^2}$
that is, d^* is a differential op. s.t.
$\forall\ \alpha, \beta \in \Omega^\bullet_c(X)$

$$\langle d^*\alpha, \beta\rangle_{L^2} = \langle \alpha, d\beta\rangle$$

$$d^* : \Omega^\bullet(X) \to \Omega^{\bullet-1}(X)$$

Recall ~~that~~ : $d = \sum_j e^j \wedge \nabla_{e_j}^{\Lambda^\bullet T^*X}$

$(e^j)^* = \iota_{e_j}$ pointwise, w.r.t $g^{\Lambda^\bullet T^*X}$

Claim : $d^* = -\sum_j \iota_{e_j} \nabla_{e_j}^{\Lambda^\bullet T^*X}$

$$D = d + d^* = \sum_j \underbrace{(e^j - \iota_{e_j})}_{c(e_j)} \nabla_{e_j}^{\Lambda^\bullet T^*X}$$ Dirac op on $\Omega^\bullet(X)$

"pf": $d^* = \sum_j (\nabla_{e_j}^{\Lambda^\bullet T^*X})^* (e^j)^*$

$= \sum_j (\nabla_{e_j}^{\Lambda^\bullet T^*X})^* \iota_{e_j}$

$= \underbrace{\sum_j [\nabla_{e_j}^{\Lambda^\bullet T^*X}, \iota_{e_j}]}_{\iota_{\sum_j \nabla_{e_j}^{TX} e_j}} + \sum_j \iota_{e_j} (\nabla_{e_j}^{\Lambda^\bullet T^*X})^*$

$$\int_X \langle \alpha, \nabla_W \beta \rangle \, dV(x) = \underbrace{\int_X W \langle \alpha, \beta \rangle \, dV(x)}_{\int_X L_W(\langle \alpha, \beta \rangle dV(x)) - \int_X \langle \alpha, \beta \rangle L_W dV(x)} - \int_X \langle \nabla_W^* \alpha, \beta \rangle \, dV(x)$$

$$(\nabla_{e_j}^{\Lambda^\bullet T^*X})^* = -\nabla_{e_j}^{\Lambda^\bullet T^*X} - \underbrace{\frac{L_{e_j} dV(x)}{dV(x)}}_{\langle e_j, \sum_k \nabla_{e_k}^{TX} e_k \rangle}$$

Rmk : $d^* d|_{\Omega^0(X) = C^\infty(X)}$ is called Beltrami Laplacian

HW 4.4

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